Inequalities Related to L^p Norm

Chen Xing

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Here, I summarize the proof for Hölder's inequality, Minkowski's inequality and monotonicity of L^p norms in finite positive measure spaces. The main reference is Stein's Functional Analysis [1] Chapter 1.

1 Hölder's inequality

Theorem 1.1. *Suppose* $0 < p < \infty$ *and* $1 < q < \infty$ *are conjugate exponents. If* $f \in L^p$ *and* $g \in L^q$*, then* $fg \in L^1$ *and*

$$||fg||_{L^1} \leq ||f||_{L^p} ||g||_{L^q}. \tag{1.1}$$

Proof. The idea is the following:

- Use generalized AM-GM Inequality.
- Normalized f, g and apply AM-GM $\leq$.

Recall that Arithmetic-Geometric mean inequality (4.1): if $A, B \geq 0$ and $0 \leq \theta < 1$, then

$$A^\theta B^{1-\theta} \leq \theta A + (1-\theta) B. \tag{1.2}$$

WLOG, assume neither f nor g vanish. By replacing f by $\frac{f}{||f||_{L^p}}$ and g by $\frac{g}{||g||_{L^q}}$, we may also assume

$$||f||_{L^p} = ||g||_{L^q} = 1.$$

Suffices to show

$$||fg||_{L^1} \leq 1.$$

Set $A = |f|^p, B = |g|^q, \theta = \frac{1}{p}$, then apply AM-GM $\leq$, we have,

$$|fg| \leq \frac{1}{p} |f|^p + \frac{1}{q} |g|^q$$

Integrating both sides,

$$||fg||_{L^1} \leq \frac{1}{p}||f||^p + \frac{1}{q}||g||^q = \frac{1}{p} + \frac{1}{q} = 1.$$

Moreover, by the equality condition in AM-GM $\leq$, we know, Hölder $\leq$ (1.2) with equality holds if

$$\frac{|f(x|)^p}{||f(x)||_p^p} = \frac{|g(x)|^q}{||g(x)||_q^q}.$$

□

Remark 1.2. *Another approach is to use Young's inequality: for nonnegative a, b,*

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}, \quad \textit{where } p, q \textit{ are conjugate exponents.}$$

Above inequality with equality holds iff $a^p = b^q$. Proof is done by first normalizing f, g, setting $a = |f|, b = |g|$ and integrating.

2 Minkowski's inequality

Motivation: It gives us the **triangle inequality** in L^p space, where $p \geq 1$. However, when $0 < p < 1$, we have a quasi-triangle inequality: $||f+g||_{L^p} \leq C_p(||f||_{L^p} + ||g||_{L^p})$. See [2].

Theorem 2.1. *If $1 \leq p < \infty$, and $f, g \in L^p$, then $f + g \in L^p$ and*

$$||f + g||_{L^p} \leq ||f||_{L^p} + ||g||_{L^p} \tag{2.1}$$

Proof. The idea is the following:

- Use Cr-inequality to show L^p space is closed under addition.
- Write $|f + g|^p = |f + g| \, |f + g|^{p-1}$, then integrating and applying Hölder.

When $p = 1$, (2.1) is obtained by integrating

$$|f + g| \leq |f| + |g| \, .$$

When $1 < p < \infty$, use Cr inequality (4.3),

$$|f + g|^p \leq 2^{r-1}(|f|^p + |g|^p),$$

which implies $f + g \in L^p$. Now, since $(p-1)q = p$,

$$\begin{aligned}
\int |f+g|^p \, d\mu &\leq \int |f+g|^{p-1} \, |f+g| \, d\mu \\
&\leq \int |f| \, |f+g|^{p-1} \, d\mu + \int |g| \, |f+g|^{p-1} \, d\mu \\
&\leq \left[\left(\int |f|^p \, d\mu \right)^{1/p} + \left(\int |g|^p \, d\mu \right)^{1/p} \right] \left(\int |f+g|^{(p-1)q} \, d\mu \right)^{\frac{1}{(p-1)q}} \quad \text{(by Hölder)} \\
&= \Big(||f||_{L^p} + ||g||_{L^p} \Big) \left(\int |f+g|^p \, d\mu \right)^{1-\frac{1}{p}} \\
&= \Big(||f||_{L^p} + ||g||_{L^p} \Big) \frac{||f+g||_p^p}{||f+g||_{L^p}}
\end{aligned}$$

Re-ranging the terms, proof is complete.

□

Remark 2.2. *Minkowski's $\leq$ with equality for $1 < p < \infty$ if and only if $f = \lambda g$, for some $\lambda \geq 0$ or $g = 0$. That is, f and g are positively linearly dependent. Another proof is to introduce a convex function, see STOR635 HW2 Q2.*

3 Monotonicity for L^p Norms

Theorem 3.1. *If Ω has **finite positive measure**, and $p_0 \leq p_1$, then $L^{p_1} \subset L^{p_0}$, and*

$$\frac{1}{\mu(\Omega)^{1/p_0}} ||f||_{p_0} \leq \frac{1}{\mu(\Omega)^{1/p_1}} ||f||_{p_1} \tag{3.1}$$

Proof. The idea is:

- Write $|f|^{p_0} = |f|^{p_0} \cdot 1$, then apply Hölder.

WOLG, assume $\mu(\Omega) = 1$. Check: $||f||_{p_0} \leq ||f||_{p_1}$. Since $\frac{p_1}{p_0} \geq 1$,

$$\begin{aligned}
||f||_{p_0}^{p_0} &= \int (|f|^{p_0} \cdot 1) \, d\mu \leq \left(\int |f|^{p_0 \cdot \frac{p_1}{p_0}} \, d\mu \right)^{\frac{p_0}{p_1}} \\
&= \left(||f||_{p_1}^{p_1} \right)^{\frac{p_0}{p_1}} = ||f||_{p_1}^{p_0},
\end{aligned}$$

which completes the proof. □

Remark 3.2. *The assumption "finite positive measure space" is necessary.*

Theorem 3.3. *Suppose $f \in L^\infty$ is supported on a set of finite measure. Then $f \in L^p$ for all $p < \infty$, and*

$$||f||_{L^p} \longrightarrow ||f||_{L^\infty}, \quad \text{as } p \longrightarrow \infty. \tag{3.2}$$

Proof. Let E be a measurable set of Ω with $\mu(E) < \infty$ so that f vanishes on E^c. If $\mu(E) = 0$, statement holds trivially. Otherwise,

$$||f||_{L^p} = \left(\int_E |f|^p d\mu \right)^{1/p} \leq \left(\int_E ||f||_\infty^p d\mu \right)^{1/p} = ||f||_\infty \mu(E)^{1/p},$$

which implies

$$\limsup_{p\to\infty} ||f||_{L^p} \leq ||f||_\infty.$$

On the other hand, given $\epsilon > 0$,

$$\mu(A) := \mu\{x : f(x) > ||f||_\infty - \epsilon\} > \delta, \quad \text{for some } \delta > 0.$$

Hence,

$$\int_\Omega |f|^p d\mu \geq \int_\Omega |f|^p \mathbf{1}_A d\mu \geq (||f||_\infty - \epsilon)^p \, \delta,$$

which implies

$$\liminf_{p\to\infty} ||f||_{L^p} \geq ||f||_\infty - \epsilon,$$

Letting $\epsilon \downarrow 0$, the proof is complete.

□

4 Appendix

Theorem 4.1 (**Generalized AM-GM Inequality**). *If* $A, B \geq 0$ *and* $0 \leq \theta < 1$, *then*

$$A^\theta B^{1-\theta} \leq \theta A + (1-\theta)B.$$

Proof. If $B = 0$, the inequality holds trivially. Assume $B \neq 0$, and replace A by AB. Then suffices to show

$$A^\theta \leq \theta A + (1-\theta) \tag{4.1}$$

Define $f(x) = x^\theta - \theta x - 1 + \theta$. Clearly, f attains a maximum at $x = 1$ and $f(1) = 0$. Hence $f(A) \leq 0$, which implies (4.1) holds. □

Remark 4.2. *AM-GM Inequality* (4.1) *with equality holds when* $A = B$.

Theorem 4.3 (**Cr-Inequality**). $\mathbf{E}|X+Y|^r \leq C_r(\mathbf{E}|X|^r + \mathbf{E}|Y|^r)$, *where*

$$C_r = \begin{cases} 1, & \text{if } 0 < r \leq 1 \\ 2^{r-1}, & \text{if } r \geq 1 \end{cases}$$

Proof. If $r \geq 1$, then $x \to |x|^r$ is convex. Then

$$|\frac{1}{2}(X+Y)|^r \leq \frac{1}{2}|X|^r + \frac{1}{2}|Y|^r$$

Taking expectation, proof is done.

If $0 < r < 1$, then $z^{1/r} \leq z$, $\forall\ 0 < z < 1$. Then

$$\left(\frac{|X|^r}{|X|^r + |Y|^r}\right)^{1/r} + \left(\frac{|Y|^r}{|X|^r + |Y|^r}\right)^{1/r} \leq \left(\frac{|X|^r}{|X|^r + |Y|^r}\right) + \left(\frac{|Y|^r}{|X|^r + |Y|^r}\right) = 1$$

After re-ranging, we get $|X+Y|^r \leq |X|^r + |Y|^r$. Taking expectation, we done. □

References

[1] Elias M. Stein and Rami Shakarchi, *Functional analysis: Introduction to further topics in analysis*, Princeton University Press, Sep 2011.

[2] Terence Tao, *An epsilon of room, ii: Pages from year three of a mathematical blog*, 2010.